Possible strategy on how to use Yang Li's ideas to produce the SLag fibration in codimension 2

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The final goal is to solve Kontsevich and Soibelman's conjecture [KS01, Conjecture 1 and 2], that is Calabi Yau fibers X_t near the maximally degenerating limit should admit $\omega_{CY,t}$-Lagrangian fibrations which are special with respect to Ω_t. For this we aim to produce a stronger version of [Li23, Theorem 4.11] (assuming the comparison property he obtains a SLag fibration where 99% of the CY-measure is found) using a similar strategy but incorporating A'Campo spaces. Throughout we interchange curvatures/forms and their potentials.

The strategy relies on showing an analogue of [Li22, Proposition 6.3] but in the submaximal locus, i.e. $\omega_{CY,t}$ is C^∞ close to a SLag fibration $(X_t, dd^c(\phi_0 \circ \overline{Log}_{\mathcal{X}}))$ over the base in codimension 2. Where

- ϕ_0 is the solution to the NA Monge Ampère equation on X^{an}, which exists by [BFJ15].
- $\overline{Log}_{\mathcal{X}}$ is the symplectic connection defined in [FdBP23] (this replaces the locally defined $Log_{\mathcal{X}}$ which maps to X^{an} used by Li).

Then by Zhang's result [Zha17, Section 4] one would get the desired SLag fibrations.

To show that $\omega_{CY,t}$ is C^∞ close to a SLag fibration $(X_t, dd^c(\phi_0 \circ \overline{Log}_{\mathcal{X}}))$, we wish to use the same idea as [Li23]. That is

1. Show that they are C^0 **close**.
2. Show a **stronger version of Vilsmeier**'s [Vil21, Theorem 1.1] (also in [Li22, Proposition 5.4]). His result states that assuming the NA MA-real MA comparison (see Definition [Li22, Definition 5.6]), that is for some model $\mathcal{X}$ we have that ϕ_0 factors through $r_{\mathcal{X}}$ on the preimage of the maximal open faces, then the pushforward through $r_{\mathcal{X}}$ of the NA MA measure gives a multiple of the real MA measure. To do: Extend Vilsmeier's result to the submaximal faces.
3. Show that the **NA MA-real MA comparison property holds** both in submaximal and maximal faces (modulo codimension 2). Problem/To do: The comparison property has not been proven in neither case.
4. If the above hold then the **regularity results** from elliptic PDE theory (see Savin's theorem [Li23, Theorem 2.8]) give that C^0 close implies C^∞ close, and then we would be done.

Let's look these steps in more detail.

1. To show that they are C^0 **close** we would want to somehow cleverly combine [Li23, Section 4] and [FdBP23]. One could construct some NA FS-metric approximation (maybe passing to the universal cover?) as in [Li23, Lemma 4.1] and the main problem probably lies in controlling the dimension of the singular locus of the regularization (compare with [Li23, Proposition 4.2]). We might also need some variation of Vilsmeier's result in this part to show some control of the CY-measure which later would allow us to apply some Skoda inequality result in order to find a stability estimate (compare with [Li23, Theorem 2.6])

2. In order to **extend Vilsmeier**'s result one first needs

   (a) An **affine integral structure** to even be able to formulate the real MA equation.

   (b) Prove the result itself.

   Let's look at them separately:

   (a) It is known that $\mathrm{Sk}(X)$ carries a canonical piecewise integral affine structure given by its embedding onto X^{an} ([MN15, 3.2]).

   Question 0.1. Is there a **preferred integral affine structure** on $\mathrm{Sk}(X)$ in codimension 2?

      i. One possibility is to create it using the construction from the D-Branes and Mirror Symmetry book.

      ii. Another possible candidate would be to use the affine structure from [NXY19, Proposition 5.4], they show that for a good minimal dlt model $\mathcal{X}$ one has that $r_{\mathcal{X}} : X^{an} \to \mathrm{Sk}(X)\backslash Z$ is a non-archimedean Lagrangian fibration, with Z the union of faces of codimension ≥ 2 in $\mathrm{Sk}(X)$. This produces an integral affine structure on $\mathrm{Sk}(X)\backslash Z$ which is compatible with the aforementioned piecewise affine structure, that is they give rise to the same piecewise integral affine functions on $\mathrm{Sk}(X)\backslash Z$.

      Question 0.2. Which model do we select? Does the retraction map $r_{\mathcal{X}}$ depend on the good minimal dlt model in codimension 2?

      We know that the retraction map does depend on the model [NXY19, §2.6], but it might be possible that it does not in codimension 2. Recall that dlt minimal models are related to each other by flops [Kaw08], it would be sufficient then to prove that flops do not change $r_{\mathcal{X}}$ in codimension 2. Problem: Flops have a very abstract definition it might be difficult to find a strategy to tackle this problem. Nevertheless some of the effects of blow-ups on the essential skeleton are known already and it points towards a positive answer [MN15, §3.2].
      Example: For degenerations of K3 surfaces dlt good minimal models or equivalently Kulikov models are related by Type 0, I and II modifications [FM83, pp. 12-15]. As seen in [NXY19, Example 2.7], both Type 0 and type II modifications do not alter $r_{\mathcal{X}}$ but Type I degenerations do. Nevertheless they only change them on a vertex of the essential skeleton (flipping a curve from a component to an other means that points specializing to that curve will be now mapped to a different vertex).
      To do: Look for other counterexamples of differing retraction maps For example in [MPS21] and [BGMPS16, Appendix] (both require non-trivial toric geometry).

   (b) To do: Understand Vilsmeier's proof and see how far one can mimic it.

A possibility is that ϕ_0 does not define a solution to the real MA. Problem: A global solution to the real MA equation in a segment might not have a global solution extending it to $\mathbb{S}^1$, since there is no convex function defined in $\mathbb{S}^1$ so one runs into problems in the degenerating elliptic curve case. Could one replace it by some multivaluated modification of ϕ_0? This does not seem to be an a priori obstruction when we go onto the degenerating K3 surface case when one takes out some points out of the sphere. Hence this might not be an obstruction in the higher dimensional case after all.

Remark 0.3. One might not have to worry so much about ϕ_0 since in the end the main interest is in $dd^c\phi_0$ so one might be able to only need some weaker arguments.

3. For the **comparison property** Li points out a couple of directions [Li22, §5.6].
   - Javier had the idea that the geometry arising from the NA SYZ fibration (from [NXY19]) could be useful to tackle this problem.
   - Probably worth reading carefully: [MPS21].
   - Not every solution of an NA MA equation satisfies the comparison property as seen in [BGMPS16, Appendix], but this counterexample is given by atomic measures of defined by the MA on model functions so it seems far from what we are looking for. Also one may construct a bigger model for which it holds.
4. To use the **regularity results** for the solution of the MA equation ϕ_0 we would like to have more information about its smooth locus on $\mathrm{Sk}(X)$. This is a problem since little is known about ϕ_0 apart from continuity and being semipositive.
   - Li deals with this problem by ignoring the singular locus which has null measure. Since his result only deals with 99% of the CY-measure, this is enough for the "weak" metric SYZ conjecture.
   - By [Li23, §2.5] the singular set has $(n-1)$ Hausdorff measure zero.

Other possible ideas:

1'. Understand Zhang's construction [Zha17] to find other ways to produce families of SLag fibrations.

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